6. Sedov, L.I., Propagation of strong waves of discontinuity. PMM Vol. 10, №2, 1946.
7. Gradshtein, I.S. and Ryzhik, I.M., Tables of Integrals, Sums, Series and Productes. Moscow, Fizmatgiz, 1962.

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GROUP CLASSIFICATION OF EQUATIONS OF HYDRODYNAMICS OF A PERFECT FLUID

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Equations of perfect fluid hydrodynamics are classified with respect to the Coriolis parameter, and all essentially different solutions of rank one are indicated.

1. Statement of the problem. Let x and y be Cartesian coordinates, u and v the velocity components along the x and y axes, respectively, and p the pressure; the density ρ is assumed constant and equal unity. We consider systems of equations of the form

$$uu_x + vu_y - lv = -p_x, \quad uv_x + vv_y + lu = -p_y, \quad u_x + v_y = 0 \qquad (1.1)$$

in which the parameter $l(y)$ can be an arbitrary function of y. For an arbitrary $l(y)$ system (1.1) admits a certain group of transformations G. The special forms of function $l(y)$ for which the fundamental group admitted by system (1.1) is wider than G are to be determined.

Equations (1.1) are encountered in meteorological problems in which the terms lu and lv represent components of acceleration produced by the Coriolis force owing to the rotation of Earth, and $l(y)$ is the Coriolis parameter. For $l = 0$ system (1.1) coincides with that of the usual equations of hydrodynamics of a perfect fluid. The determination of the group for this case is given in [1] on the assumption of unsteady flow.

Besides the determination of the group of transformations we shall derive solutions of rank one, i.e. such which reduce their derivation to the integration of ordinary differential equations. Some of these solutions were obtained earlier, for instance, in [2] solutions with spiral streamlines are indicated. In the present paper the problem of group classification of system (1.1) is solved, optimal systems of one-parameter subgroups are determined, and all essentially different solutions are indicated. Since the required mechanism of group analysis is presented in [3], many intermediate computations are omitted.

2. Classification of equations. To calculate the coordinates of the infinitesimal operator of the group admitted by system (1.1) it is necessary to write out the so-called defining equations and to solve these.

1) For any arbitrary function $l(y)$ the basic operators of the related Lie algebra are of the form

$$X_1 = \partial / \partial p, \qquad X_2 = \partial / \partial x \qquad (2.1)$$

The analysis of determining equations for other forms of function $l(y)$ yields the following results:

2) if $l = y^{m-1}$ $(m \neq 1)$, operator

$$X_3^1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + mu\frac{\partial}{\partial u} + mv\frac{\partial}{\partial v} + 2mp\frac{\partial}{\partial p} \tag{2.2}$$

is added to the operators(2.1);

3) if $l = e^{mv}$ $(m \neq 0)$, operator

$$X_3^2 = \frac{\partial}{\partial y} + mu\frac{\partial}{\partial u} + mv\frac{\partial}{\partial v} + 2mp\frac{\partial}{\partial p} \tag{2.3}$$

is added to operators (2.1);

4) if $l = 1$, the fundamental group is generated by five operators with three operators

$$X_3^3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + 2p\frac{\partial}{\partial p} \tag{2.4}$$

$$X_4 = \frac{\partial}{\partial y}, \qquad X_5 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}$$

added to (2.1), and

5) if $l = 0$, the group is generated by six operators with operators

$$X_3^4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad X_4 = \frac{\partial}{\partial y}$$

$$X_5 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v} \tag{2.5}$$

$$X_6 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + 2p\frac{\partial}{\partial p}$$

added to (2.1).

3. Solution for $l \neq$ const. Let us construct the essentially different solutions for the first three cases. For this it is necessary to determine the optimum systems of one-parameter subgroups and integrate the derived equations. The following notation is used henceforth: u_0, v_0 and p_0 are arbitrary constants, and f is an arbitrary function of its argument.

3.1. If $l(y)$ is an arbitrary function, the optimum system is generated by operators (2.1). The subgroup with operator X_1 is eliminated, since for it the necessary condition of the existence of an invariant solution is not satisfied, while the subgroup with operator X_2 provides a solution dependent on y. The substitution of these into Eqs. (1.1) yields the system

$$vu' - lv = 0, \qquad vv' + lu = -p', \qquad v' = 0$$

where the prime denotes differentiation. From the last equation we have $v = v_0$. For $v_0 \neq 0$ the integration yields

$$u = u_0 + \lambda(y), \qquad p = p_0 - {}^1/_2\,\lambda^2 + u_0\lambda \qquad (\lambda'(y) = l(y))$$

If, however, $v_0 = 0$, then

$$u = u_0 + f(y), \qquad p = p_0 + \int l(y)\,[u_0 + f(y)]\,dy$$

In the first case the streamlines are defined by formula

$$u_0 y - v_0 x + \int \lambda(y)\,dy = \text{const}$$

and in the second the streamlines are in the form of straight lines parallel to the x-axis.

3.2. For $l = y^{m-1}$ the optimum system is generated by operators X_1, X_3^1 and $X_2 + \alpha X_1$. The solution for the subgroup with operator X_3^1 is of the form

$$u = x^m U(\xi), \quad v = x^m V(\xi), \quad p = x^{2m} P(\xi), \; \xi = y / x$$

The last subgroup yields a solution of the form

$$u = U(y), \quad v = V(y), \quad p = \alpha x + P(y)$$

Substituting into (1.1) and integrating, we obtain the following solution:

$$u = u_0 + \frac{1}{m} y^m - \frac{\alpha}{v_0} y, \qquad v = v_0$$

$$p = p_0 + \alpha x - \frac{1}{2m^2} y^{2m} + \frac{\alpha}{v_0 (m+1)} y^{m+1} - \frac{u_0}{m} y^m$$

It is assumed that in this case $m \neq 0$ and $m \neq -1$.

In meteorological problems the Coriolis parameter is often approximated by a linear function, which corresponds to $m = 2$. In this case the streamlines are cubic parabolas, and the solution may be treated as defining a "crest" type of flow. Meteorological observations show that such flows result in the formation of fronts with abrupt change of weather.

If $m = 0$, the solution is of the form

$$u = u_0 + \ln y - \frac{\alpha}{v_0} y, \qquad v = v_0$$

$$p = p_0 + \alpha x - \frac{1}{2} \ln^2 y + \frac{\alpha}{v_0} y - u_0 \ln y$$

and for $m = -1$

$$u = u_0 - \frac{1}{y} - \frac{\alpha}{v_0} y, \qquad v = v_0$$

$$p = p_0 + \alpha x - \frac{1}{2y^2} + \frac{\alpha}{v_0} \ln y + \frac{u_0}{y}$$

3.3. Let us consider the group for which $l = e^{my}$. The optimum system of one-parametric subgroups is generated by operators $\alpha X_1 + X_2$ and $X_3^2 + \alpha X_2$.

The first subgroup yields solutions of the form

$$u = U(y), \quad v = V(y), \quad p = \alpha x + P(y)$$

Substituting into system (1.1) and integrating, we obtain

$$u = u_0 + \frac{1}{m} e^{my} - \frac{\alpha}{v_0} y, \qquad v = v_0$$

$$p = p_0 + \alpha x + \frac{1}{m^2}\left[\frac{\alpha}{v_0}(my - 1) - \frac{1}{2} - m u_0\right] e^{my}$$

which is similar to that for the crest type of flow.

The solution for the second subgroup is of the form

$$u = e^{my} U(\xi), \quad v = e^{my} V(\xi), \quad p = e^{2my} P(\xi), \quad \xi = y - \alpha x$$

and the unknown functions satisfy the following system of equations:

$$UU' + V(mU - \alpha U') - V = -P'$$

$$UV' + V(mV - \alpha V') + U = -2mP + \alpha P'$$
$$U' + mV - \alpha V' = 0$$

One of the solutions of this system is

$$U = a + v_0 e^{k\xi}, \quad V = v_0 e^{k\xi}, \quad P = bv_0 e^{k\xi} - a/2m$$

$$a = \frac{(\alpha - 1)^2}{m(\alpha^2 - 2\alpha - 1)}, \quad b = \frac{1 - \alpha^2}{m(\alpha^2 - 2\alpha - 1)}, \quad k = \frac{m}{\alpha - 1}$$

The streamlines ψ = const are specified (correct to within the shift along the x-axis) by the formula

$$x = y - \frac{\alpha - 1}{m\alpha} \ln\left(\text{const} - \frac{a}{m} e^{my}\right)$$

and represent a set of curves tending to a straight line. This solution can simulate the flow in a cumulative stream.

4. Solutions for constant Coriolis parameter. Before proceeding with the construction of solution for $l = 1$ and $l = 0$, we note that the input system (1.1) for a constant Coriolis parameter can be replaced by its equivalent system by the substitution of the vortex $\omega = u_y - v_x$ for one of the unknown functions. For $\omega = 0$ system (1.1) is equivalent to the Cauchy-Riemann equations $u_y - v_x = 0$ and $u_x + v_y = 0$, which admits an infinite group of transformations. Because of this we shall subsequently seek only such solutions in which the vortex is nonzero.

Below it will be necessary to use the input equations in polar coordinates

$$x = r\cos\vartheta, \quad y = r\sin\vartheta, \quad u = w\cos\varphi, \quad v = w\sin\varphi$$

The expressions for the vortex and the stream functions in these coordinates are

$$\omega = \cos(\varphi - \vartheta)\left(\frac{1}{r} w_\vartheta - w\varphi_r\right) - \sin(\varphi - \vartheta)\left(w_r + \frac{w}{r}\varphi_\vartheta\right)$$

$$\psi_r = -w\sin(\varphi - \vartheta), \qquad r^{-1}\psi_\vartheta = w\cos(\varphi - \vartheta)$$

The arbitrary constants of integration of equations in polar coordinates are denoted by w_0, p_0 and φ_0.

4.1. Let us consider the solutions for $l = 1$. The optimum system is generated by seven operators

$$X_1,\ X_2,\ X_3^3,\ X_5,\ X_1 + X_2,\ X_2 + X_5,\ X_3^3 + \alpha X_5 \tag{4.1}$$

Let us write Eqs. (1.1) in polar coordinates. By combining these and taking into account trigonometric identities, we can obtain the following system:

$$\frac{w^2}{r}\varphi_\vartheta + w\sin(\varphi - \vartheta) = p_r, \quad w^2\varphi_r + w\cos(\varphi - \vartheta) = -\frac{1}{r}p_\vartheta$$
$$\cos(\varphi - \vartheta)\left(w_r + \frac{w}{r}\varphi_\vartheta\right) - \sin(\varphi - \vartheta)\left(w\varphi_r - \frac{1}{r}w_\vartheta\right) = 0 \tag{4.2}$$

In the same variables the operators X_3^3 and X_5 assume the form

$$X_3^3 = r\frac{\partial}{\partial r} + w\frac{\partial}{\partial w} + 2p\frac{\partial}{\partial p}, \qquad X_5 = \frac{\partial}{\partial\vartheta} + \frac{\partial}{\partial\varphi}$$

4.1.1. The subgroup with operator X_1 does not yield invariant solutions, since the necessary condition for the existence of such solutions is not satisfied.

4.1.2. The subgroup with operator X_2 yields solutions which depend only on y.

Integrating Eqs. (1.1) we obtain

$$u = u_0 + y,\ v = v_0,\ p = p_0 - {}^1/_2\, y^2 - u_0 y, \quad \text{if} \quad v_0 \neq 0$$
$$u = -f'(y),\ v = 0,\ p = f(y), \quad \text{if} \quad v_0 = 0 \qquad (4.3)$$

In the first case the vortex $\omega = 1$ and the streamlines are represented by a set of parabolas. In the second, $\omega = -f''$ and the streamlines are straight lines parallel to the x-axis.

4.1.3. For the subgroup with operator X_3^3 we seek the solution in the form $w = r\,W(\vartheta)$, $\varphi = \Phi(\vartheta)$ and $p = r^2 P(\vartheta)$. Substituting into (4.2), we obtain equations

$$W^2\Phi' + W \sin(\Phi - \vartheta) = 2P, \quad W\cos(\Phi - \vartheta) = -P'$$
$$\cos(\Phi - \vartheta)\, W(1 + \Phi') + \sin(\Phi - \vartheta)\, W' = 0$$

In integrating this system we must consider two cases, depending on whether Φ' is zero or not. For $\Phi' = 0$ the solution is of the form

$$w = w_0\, r \sin(\vartheta - \varphi_0), \quad \varphi = \varphi_0$$
$$p = -\frac{w_0}{2} r^2 \sin^2(\vartheta - \varphi_0)$$

It can be shown that $\omega = w_0$. This solution yields streamlines which are parallel straight lines at angle φ_0 to the x_0-axis. Since the direction of coordinate axes was not specified, the x-axis can be made to coincide with that of the velocity vector, i. e. we can set $\varphi_0 = 0$. The absolute value of the velocity vector is proportional to y.

The second case, in which $\Phi' \neq 0$ yields a solution which can be written in the parametric form

$$w = r\sqrt{\frac{2p_0}{z+1}}, \quad \varphi = \vartheta + \arcsin\left(w_0 \frac{z+2}{\sqrt{z+1}}\right)$$
$$p = r^2\left(p_0 + w_0 \frac{z+2}{z+1}\sqrt{\frac{p_0}{2}}\right) \qquad (4.4)$$
$$\vartheta = \varphi_0 + \frac{1}{2}\arcsin\frac{(1 - 2w_0^2)(z+1) - 2w_0^2}{(z+1)\sqrt{1 - 4w_0^2}}$$

For the vortex and the stream function we have

$$\omega = -\frac{\sqrt{2p_0}}{w_0}, \quad \psi = \psi_0 - \frac{w_0}{2}\sqrt{2p_0}\, r^2 \frac{z+2}{z+1} = -\psi_0 + ar^2 + br^2 \sin(2\vartheta - 2\varphi_0)$$
$$a = \frac{1}{2w_0}\sqrt{\frac{p_0}{2}}, \quad b = \sqrt{\frac{p_0}{2}}\sqrt{\frac{1}{4w_0} - 1}$$

By a suitable selection of the direction of coordinate axes it is possible to obtain φ_0 equal zero. The streamlines are then defined by formula

$$a(x^2 + y^2) - 2bxy + \text{const} = 0$$

i. e. they are represented by a set of ellipses in a system of coordinates turned by 45°.

4.1.4. The subgroup with operator X_5 yields solution of the form $w = W(r)$, $\varphi = \vartheta + \Phi(r)$ and $p = P(r)$. Substituting into (4.2) we obtain the system

$$\frac{1}{r}W^2 + W\sin\Phi = P', \qquad W^2\Phi' + W\cos\Phi = 0$$
$$\cos\Phi\left(W' + \frac{1}{r}W\right) - \sin\Phi W\Phi' = 0 \qquad (4.5)$$

The last equation yields the first integral $rW\cos\Phi = w_0$. We can now write the general solution

$$w = \frac{w_0}{r}\sqrt{1+\left(\varphi_0-\frac{r^2}{2w_0}\right)^2}, \qquad \varphi = \vartheta + \operatorname{arctg}\left(\varphi_0 - \frac{r^2}{2w_0}\right)$$

$$p = p_0 - \frac{r^2}{8} - (1+\varphi_0^2)\frac{w_0^2}{2r^2}$$

The streamlines $\vartheta = \varphi_0 \ln - {}^1/_4\, r^2 / w_0 + \text{const}$ are represented by a set of spiral lines with a source ($w_0 > 0$) or sink ($w_0 < 0$) at the coordinate origin. For $w_0\varphi_0 < 0$ the angle ϑ along the streamline monotonically varies, while for $w_0\varphi_0 > 0$ the monotonicity breaks down. It can be shown by direct calculation that for such flow the vortex is constant and equal unity. A similar solution was previously derived in meteorology [4].

4.1.5. For the subgroup with operator $X_1 + X_2$ the solution is sought in the form $u = U(y)$, $v = V(y)$ and $p = x + P(y)$. Substitution into Eqs. (1.1) and integration yields

$$u = u_0 + \frac{v_0 - 1}{v_0} y, \qquad v = v_0$$

$$p = p_0 + x + u_0 y - \frac{v_0 - 1}{2v_0} y^2, \qquad \omega = \frac{v_0 - 1}{2v_0}$$

If $v_0 \neq 1$, the vortex is nonzero, and the streamlines are parabolas with their axis parallel to the x-axis. Note that in these formulas $v_0 \neq 0$, since otherwise system (1.1) becomes inconsistent.

4.1.6. For the subgroup with operator $X_1 + X_5$ the solution is of the form $w = W(r)$, $\varphi = \vartheta + \Phi(r)$ and $p = \vartheta + P(r)$.

The unknown functions are determined by system (4.5) in which $-1/r$ is to be substituted into the right-hand side of the second equation. Integration of obtained equations yields the solution

$$w = \frac{w_0}{r}\sqrt{1+\left(\varphi_0 - \frac{w_0+1}{2w_0^2} r^2\right)^2}$$

$$\varphi = \vartheta + \operatorname{arctg}\left(\varphi_0 - \frac{w_0+1}{2w_0^2} r^2\right)$$

$$p = p_0 + \vartheta - \left(1 - \frac{1}{w_0^2}\right)\frac{r^2}{8} - (1+\varphi_0^2)\frac{w_0^2}{2r^2} - \varphi_0 \ln r$$

The solution is similar to that derived in 4.1.4, and the vortex $w = 1 + 1/w_0$ is throughout nonzero.

4.1.7. The last subgroup with operator $X_3^3 + \alpha X_5$ generates a solution of the form $w = rW(\xi)$, $\varphi = \vartheta + \Phi(\xi)$ and $p = r^2 P(\xi)$, where $\xi = re^{-\vartheta}$.

4.2. Let us pass to the derivation of solution for $l = 0$. Equations (1.1) in polar coordinates are of the form

$$\frac{w^2}{r}\varphi_\vartheta = p_r, \qquad w^2\varphi_r = -\frac{1}{r}p_\vartheta \tag{4.6}$$

$$\cos(\varphi - \vartheta)\left(w_r + \frac{w}{r}\varphi_\vartheta\right) - \sin(\varphi - \vartheta)\left(w\varphi_r - \frac{1}{r}w_\vartheta\right) = 0$$

In certain instances the last equation will be presented in a somewhat different form by the substitution for the derivatives of φ of their expressions in the first two equations. Note that in polar coordinates operators X_3^4 and X_6 are of the simpler form

$$X_3^4 = r\frac{\partial}{\partial r}, \qquad X_6 = w\frac{\partial}{\partial w} + 2p\frac{\partial}{\partial p}$$

The optimum system of one-parameter subgroups is generated by eleven operators which for convenience are divided into two classes

$$1)\ X_2,\ X_3^4,\ X_5,\ X_3^4 + X_5$$
$$2)\ X_1 + X_2,\ X_1 + X_3^4,\ X_2 + X_6,\ X_1 + X_5$$
$$X_1 + X_3^4 + X_5,\ X_3^4 + X_6,\ X_5 + X_6$$

Direct calculation shows that operators of the first class yield solutions which define vortex-free flows only, hence, in accordance with the previously stated, we restrict the analysis to operators of the second class.

4.2.1. The subgroup with operator $X_1 + X_2$ yields a trivial solution $u = u_0 - y / v_0$, $v = v_0$ and $p = p_0$. The streamlines are parabolas with their axis parallel to the x-axis. The constant v_0 is nonzero, since otherwise system (1.1) becomes inconsistent.

4.2.2. For the subgroup with operator $X_1 + X_3^4$ we seek the solution of the form $w = W(\vartheta)$, $\varphi = \Phi(\vartheta)$ and $p = \ln r + P(\vartheta)$. After substitution into (4.6) we obtain

$$W^2\Phi' = 1,\ \cos(\Phi - \vartheta)\, W\,\Phi' + \sin(\Phi - \vartheta)\, W' = 0,\ P = p_0$$

We use the first of these equations for determining W in terms of Φ and reduce the second equation to $F'' = 2(1 + F')^2 \operatorname{ctg} F$ by introducing the new function $F = \Phi - \vartheta$ and substituting F for W. The order of this equation is reduced by one by setting $F' = z(F)$. The lower order equation is integrated and yields the following dependence of F on z:

$$w_0 \sin^2 F = (z+1)\exp\left(\frac{1}{z+1}\right)$$

Let us consider z as a parameter, which reduces the derivation of solution to the determination of one quadrature. The solution itself can be presented in the parametric form

$$w = \sqrt{t}, \qquad \varphi = \vartheta + F(t), \qquad p = p_0 + \ln r$$

$$F(t) = \arcsin\left(\frac{e^{t/2}}{\sqrt{w_0 t}}\right), \qquad \vartheta = \varphi_0 - \frac{1}{2}\int\frac{dt}{\sqrt{w_0 t e^{-t} - 1}}, \qquad \left(t = \frac{1}{z+1}\right)$$

The vortex $\omega = -\sqrt{w_0} e^{-t/2}/r$ is throughout nonzero. The streamlines are represented by a set of spiral lines, and the solution exists only for $w_0 > e$.

4.2.3. The subgroup $X_2 + X_6$ generates a solution of the form $u = e^x U(y)$, $v = e^x V(y)$ and $p = e^{2x} P(y)$. The substitution of these expressions into Eqs. (1.1) yields

$$VV'' - V'^2 = 2p_0, \qquad P = p_0 \tag{4.7}$$

If $p_0 = 0$, then the solution has the form $u = -u_0 v_0 e^{x+v_0 y}$, $v = u_0 e^{x+v_0 y}$, $p = 0$ and the streamlines are straight lines $x + v_0 y = \text{const}$. But if $p_0 \neq 0$, the second of Eqs. (4.7), after single integration, yields

$$V'^2 = AV^2 - B \tag{4.8}$$

where A and B are arbitrary constants. Several cases must be considered, depending on the signs of A and B.

1) If $A = v_0^2$ and $B / v_0^2 = -p_0^2$ the solution is of the form

$$u = -v_0 p_0 e^x \ \mathrm{ch}\ (v_0 y + u_0)$$
$$v = p_0 e^x \,\mathrm{sh}\ (v_0 y + u_0), \qquad p = -{}^1/_2\ v_0^2\ p_0^2 e^{2x}$$

If $p_0 < 0$ and $v_0 > 0$, the streamlines approach the straight line $y = y_0 = -u_0 / v_0$; if, however, $p_0 > 0$ and $v_0 < 0$, the streamlines move away from the straight line $y = y_0$. Other combinations of these inequalities yield a similar flow pattern, except that the direction of the velocity vector changes to opposite.

2) When $A = v_0^2$ and $B / v_0^2 = p_0^2$, the integration of Eq. (4.8) yields the solution

$$u = -v_0 p_0 e^x \,\mathrm{sh}\ (v_0 y + u_0)$$
$$v = p_0 e^x \,\mathrm{ch} (v_0 y + u_0), \qquad p = {}^1/_2\ v_0^2 p_0^2 e^{2x}$$

The streamlines have the straight line $y_0 = -u_0/v_0$ as their vertical tangent and resemble a flow of the crest type.

3) If $A = -v_0^2$ and $B / v_0^2 = p_0^2$, the integration of (4.8) yields

$$u = -v_0 p_0 e^x \cos\ (v_0 y + u_0)$$
$$v = p_0 e^x \sin\ (v_0 y + u_0), \quad p = -{}^1/_2\ v_0^2\ p_0^2 e^{2x}$$

The solution is periodic with respect to y. The flow is divided into bands π / v_0 wide inside which the direction of the velocity vector monotonically changes to opposite along the streamlines from one boundary of the band to the other. The flow resembles that with "contact discontinuity" of equations for a compressible fluid.

The vortex is defined by formula $\omega = p_0 (v_0^2 - 1)\ e^x \sin\ (v_0 y + u_0)$. For $|\ v_0\ | \neq 1$ the vortex is nonzero. For the first two cases it is nonzero for any values of the constants.

4.2.4. For the subgroups with operator $X_1 + X_5$ the solution is sought in the form $w = W(r)$, $\varphi = \vartheta + \Phi(r)$ and $p = \vartheta + P(r)$. Substituting into (4.6) and integrating, we obtain the following solution

$$w = \frac{w_0}{r}\sqrt{1 + \left(\varphi_0 - \frac{r^2}{2w_0^2}\right)^2}, \qquad \varphi = \vartheta + \mathrm{arctg}\left(\varphi_0 - \frac{r^2}{2w_0^2}\right)$$
$$p = p_0 + \vartheta + \frac{1}{8w_0} r^2 - (1 + \varphi_0^2)\frac{w_0^2}{2r^2} - \varphi_0 \ln r$$

The constant w_0 is assumed to be nonzero, since otherwise the system of equations would be inconsistent. The vortex is determined by $\omega = 1 / w_0$, and the solution is similar to that derived in 4.1.4.

4.2.5. The solution generated by the subgroup with operator $X_1 + X_3^4 + X_5$ can be represented by $w = W(\xi)$, $\varphi = \vartheta + \Phi(\xi)$ and $p = \vartheta + P(\xi)$, where $\xi = re^{-\vartheta}$ is the new independent variable.

4.2.6. For the subgroup with operator $X_3^4 + X_6$ the solution is specified by formulas $w = rW(\vartheta)$, $\varphi = \Phi(\vartheta)$ and $p = r^2 P(\vartheta)$, where the unknown functions are determined by equations

$$P = p_0,\ W^2\Phi' = 2\,p_0,\ \cos\ (\Phi - \vartheta)\ (W^2 + 2p_0) + \sin\ (\Phi - \vartheta)\ WW' = 0$$

For $p_0 = 0$ the solution of this system is $w\, w_0 r \sin(\vartheta - \varphi_0)$, $\varphi = \varphi_0$ and $p = 0$, which in the system of coordinates turned by angle φ_0 corresponds to the Couette type flow. If $p_0 \neq 0$, the solution can be presented in the parametric form

$$w = r\sqrt{\frac{2p_0}{z+1}}, \qquad \varphi = \vartheta + \arcsin\left(w_0 \frac{z-2}{\sqrt{z+1}}\right)$$

$$p = p_0 r^2, \qquad \vartheta = \varphi_0 + \frac{1}{2}\arcsin\frac{(1-2w_0^2)(z+1)-2w_0^2}{(z+1)\sqrt{1-4w_0^2}}$$

This solution is the same as (4.4), except for the expression for pressure.

4.2.7. The last subgroup with operator $X_5 + X_6$ generates the solution $w = e^{\vartheta} W(r)$, $\varphi = \vartheta + \Phi(r)$ and $p = e^{2\vartheta} P(r)$.

BIBLIOGRAPHY

1. Buchnev, A. A., The Lie group admitted by equations of incompressible fluid. Collection: Dynamics of Continuous Media, №7, Novosibirsk, 1971.
2. Müller, E. A. und Matschat, K., Über das Anfinden von Ähnlichkeitslösungen partieller Differentialgleichungssysteme unter Benutzung von Transformationsgruppen, mit Anwendung auf Probleme der Strömungsphysik. Miszellaneen der Angewandte Mechanik, zum 60-sten Geburstag am 13 Oktober, 1960.
3. Ovsiannikov, L. V., Group Properties of Differential Equations, Novosibirsk, 1962.
4. Katkov, V. L., Exact solutions of the equation of geopotential forecast. Izv. Akad. Nauk SSSR, Ser. Fiz. Atmosfery i Okeana, Vol. 11, 1966.

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PENETRATION OF A CONE INTO A COMPRESSIBLE FLUID

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The self-similar axisymmetric problem of penetration of a blunt rigid cone into a half-space of perfect compressible fluid is considered in linear formulation.

The problem of penetration of a blunt cone into an incompressible fluid was investigated theoretically [1-4] and experimentally [3, 5]. This problem was solved for a compressible fluid in [6] on the assumption that the radius of the intersection circle between the cone and the unperturbed fluid surface inctreases, when the penetration velocity exceeds the speed of sound in the fluid (the supersonic case).

An exact analytical solution of this problem in the subsonic case is derived here with allowance for the rise of the fluid free surface in the cone neighborhood. The distribution of pressure and forces acting on the cone is presented in terms of elementary functions, and the rate of increase of the cone wetted surface radius is determined. It is shown that in the limit case of incompressible fluid the obtained results coincide with published data, while in the other limit case the derived solution coincides with that for the case of supersonic penetra-